IRREDUCIBLE COMPONENTS OF GENERIC COMPLETE INTERSECTIONS

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Newton polyhedra theory connects algebraic geometry to the geometry of convex polyhdra with integral vertices in the framework of toric geometry.

A *Laurent polynomial* P is a linear combination of monomials (possibly of negative powers). The *support* $s(P)$ of P is the set of the powers of monomials appearing in P with nonzero coefficients. The *Newton polyhedron* $\Delta(P)$ of P is the convex hull of $s(P)$. Fix k finite subsets $A_1, \dots, A_k$ in the lattice $\mathbb{Z}^n$ and consider a generic k-tuple of Laurent polynomials $P_1, \dots, P_k$ with the supports $s(P_1) = A_1, \dots, s(P_k) = A_k$. Let X be the algebraic variety defined by the system

$$P_1 = \cdots = P_k = 0 \tag{1}$$

in $(\mathbb{C}^*)^n$. Let $\Delta_1, \dots, \Delta_k$ be Newton polyhedra of $P_1, \dots, P_k$.

Definition 1. For fixed k-tuple of convex bodies $\Delta_1, \dots, \Delta_k$ in $\mathbb{R}^n$ for any nonempty subset $J \subset \{1, \dots, k\}$ we define the *defect* $d(J)$ of J to be the number $d(J) = \dim(\Delta_J) - |J|$, where $\Delta_J = \sum_{i \in J} \Delta_i$ and $|J|$ is the number of elements in J.

Definition 2. We call k-tuple $\Delta_1, \dots, \Delta_k$ of convex bodies *independent* if the defect of any nonempty subset $J \subset \{1, \dots, k\}$ is nonnegative.

Theorem (David Bernstein, 1975). *The algebraic variety $X \subset (\mathbb{C}^*)^n$ defined by a generic system (1) is nonempty if and only if the k-tuple of Newton polyhedra $\Delta_1, \dots, \Delta_k$ is independent (in the sense of Definition 2).*

According to the Newton polyhedra theory *all natural discrete* *invariants of the variety X defined by a generic system (1) depend only on* $\Delta_1, \dots, \Delta_k$ (and are independent of a choice of supports $A_1, \dots, A_k$ whose convex hulls are $\Delta_1, \dots, \Delta_k$ and of a choice of generic k-tuple of Laurent polynomials with these supports).

We compute the number of irreducible components of the algebraic variety X defined by a generic system (1). Our results (see theorem 1-3) generalize the famous Bernstein-Kouchnirenko theorem (see its statement below). This amazing theorem inspired much activity that eventually lead to the creation of the Newton polyhedra theory, of a birationally invariant version of the intersection theory for divisors [3] and of the theory of Newton-Okounkov bodies [4,5].

Let L be a real m-dimensional linear space containing a fixed discrete additive subgroup $\Lambda \subset L$ of rank m. One can defined the unique translation invariant *integral volume* on L normalized by the following condition: a m-dimensional parallepiped based on vectors $e_1, \dots, e_m \in \Lambda$ has the integral volume one if and only if vectors $e_1, \dots, e_m$ form a basis in Λ.

The *space* $\mathbb{R}^n$ *of characters* of the torus $(\mathbb{C}^*)^n$ is equipped with the *lattice* $\mathbb{Z}^n$ *of characters*, so the integral volume on $\mathbb{R}^n$ is well defined. Newton polyhedra of Laurent polynomials on the torus $(\mathbb{C}^*)^n$ belong to the space $\mathbb{R}^n$ of characters, so one can talk about the integral volume of Newton polyhedra.

Theorem (Bernstein-Koushnerenko, 1975). *For $k = n$ the variety X defined by a generic system (1) is a finite set containing $n!V(\Delta_1, \dots, \Delta_n)$ points where $V(., \dots, .)$ is the mixed volume associated with the integral volume on $\mathbb{R}^n$.*

The David Bernstein theorem follows from the Bernstein-Koushnirenko theorem and from a Minkowsky theorem state below (its proof can be found in [7]).

Theorem (Minkowsky). *A given n-tuple of convex bodies in $\mathbb{R}^n$ has the mixed volume equal to zero if and only if the n-tuple of convex bodies is dependent.*

If $k = n$ then the variety X is zero dimensional and number of its irreducible components is equal to the number of points in X. Let us drop the assumption that $k = n$. One can assume that the k-tuple of Newton polyhedra $\Delta_1, \dots, \Delta_k$ is independent: otherwise according to the David Bernstein theorem the variety X defined by a generic system (1) is empty.

Theorem 1 ([7]). *If for the k-tuple of Newton polyhedra $\Delta_1, \dots, \Delta_k$ the defect $d(J)$ of each nonempty subset $J \subset \{1, \dots, k\}$ is positive then the algebraic variety X defined by a generic system (1) is irreducible.*

The theorem 1 generalizes the following previously know result.

Theorem 1' ([2]). *If all Newton polyhedra $\Delta_1, \dots, \Delta_k$ have dimension n and $k < n$ then the algebraic variety X is irreducible.*

Our proof of the theorem 1 (see [7]) is based on toric technique, including toric resolution of singularities of toric varieties and computations of cohomologies of invariant linear bundles on toric varieties. Very similar arguments allow to compute the arithmetic genus of X. For zero dimensional varieties X it implies the Bernstein-Koushnirenko theorem (see [1,2]).

Let $J = \{i_1, \dots, i_p\} \subset \{1, \dots, k\}$ be a biggest with respect to inclusion subset among all nonempty subsets with zero defect. Denote by Δ_J the polyhedron $\Delta_J = \sum_{i\in J} \Delta_i$. Let $L_J \subset \mathbb{R}^n$ be the linear space parallel to the smallest affine subspace containing the polyhedron Δ_J. Consider the p-tuple $\Delta_{i_1}, \dots, \Delta_{i_p}$ of Newton polyhedra (where $\{i_1, \dots, i_p\} = J$). Polyhedra Δ_{i_j} for $i_j \in J$ can be shifted by parallel translation into the space L_J. Thus the mixed volume

$V(\Delta_{i_1}, \ldots, \Delta_{i_p})$ with respect to the integral volume on L_J is well defined.

Theorem 2 (**[7]**). *In the above notations the number $b_0(X)$ of the irreducible components of X is equal to $p!V(\Delta_{i_1}, \ldots, \Delta_{i_p})$.*

The theorem 2 could be easily reduced to the theorem 1 (see [7]).

Corollary. *The variety X is irreducible only in the following cases: 1) the k-tuple $\Delta_1, \ldots, \Delta_k$ of Newton polyhedra is independent (see theorem 1); 2) the number $p!V(\Delta_{i_1}, \ldots, \Delta_{i_p})$ (see theorem 2) is equal to one.*

Remark 1. Because of the corollary 1 the following question is important for us: is it possible to classify geometrically all p-tuples of integral polyhedra in p-dimensional space whose integral mixed volume multiplied by $p!$ is equal to one? The answer on this question is positive. Such classification is described in [6].

With the rational p-dimensional space $L_J \subset \mathbb{R}^n$ one can associate the subtorus T^m of dimension $m = n - p$ in the torus $(\mathbb{C}^*)^n$, defined by the following condition: $g \in T^m$ if and only if $\chi(q) = 1$ for each character χ whose power belongs to the lattice $\mathbb{Z}^n \bigcap L_J$. The embedding $\pi : T^m \to (\mathbb{C}^*)^n$ induces t he linear map $\pi^* : \mathbb{R}^n \to \mathbb{R}^m$ from the space $\mathbb{R}^n$ of characters on $(\mathbb{C}^*)^n$ to the space $\mathbb{R}^m$ of characters on T^m. With each polyhedron with integral vertices $\Delta \subset \mathbb{R}^n$ one can associate the polyhedron with integral vertices $\pi^*(\Delta) \subset \mathbb{R}^m$.

Theorem 3 (**[7]**). *In the assumptions of the theorem 2 each irreducible component of the variety X is isomorphic to a variety $Y \subset T^m$ defined by a system $Q_{q_1} = \cdots = Q_{q_m} = 0$ where $\{q_1, \ldots, q_m\} = \{1, \ldots, k\} \setminus J$ and $Q_{q_1}, \ldots, Q_{q_m}$ is a generic m-tuple of Laurent polynomials with Newton polyhedra $\pi^*(\Delta_{q_1}), \ldots, \pi^*(\Delta_{q_m})$.*

Our proof of the theorem 3 is based on a simple explicit construction (see [7]).

Remark 2. The theorem 2 computes the number $b_0(X)$ of irreducible components of X and the theorem 3 allows to compute all natural discrete invariants of each component of X (each such invariant takes the same value at all components of X). Indeed, according to the Newton polyhedra theory all natural discrete invariants of Y can be computed in terms of Newton polyhedra $\pi^*(\Delta_{q_1}), \ldots, \pi^*(\Delta_{q_m})$.

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INTEGRABILITY PROPERTIES OF NLS TYPE EQUATIONS VIA DARBOUX TRANSFORMATIONS, AND RELATED YANG-BAXTER MAPS

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Darboux transformations constitute a very important tool in the theory of integrable systems. They map trivial solutions of integrable partial differential equations to non-trivial ones and they link